Exam Quantum Physics II

Examenhal, Friday, July 7, 2006, 14:00-17:00.

Before you start, read the following:

- There are 4 problems with a total of 50 points.
- Write your name and student number on every sheet of paper.
- Write the solution of each problem on a separate sheet of paper.
- Illegible writing will be graded as incorrect.
- *Good luck!*

Problem 1 (*45 minutes; 15 points in total*)

Answer the following questions, brief and to the point:

2 pnts (a) Evaluate the commutator $[L_i, x_j]$.

2 pnts (b) Calculate the energy in eV, and the wavelength in nm, of the Balmer H-β line ($n = 4 \rightarrow n = 2$) in hydrogen. Use $\alpha = 1/137$ and $\hbar c = 200$ eV·nm. What is the color of this line?

2 pnts (c) What is the ground-state energy in eV of *protonium*, the hydrogen-like bound state of a proton and an antiproton? How large is the Bohr radius?

2 pnts (d) Sketch (no formulas!) the fine-structure of the $n = 3$ level in hydrogen. Label the levels with the appropriate quantum numbers.

2 pnts (e) Write down the Hamiltonian of the helium atom. What is the ground-state energy, in formula and calculated in eV, when the interaction between the electrons is neglected? What is the ground state in term notation ${}^{2S+1}L_J$?

2 pnts (f) Show that the ground-state electronic configuration ${}^7S_3$ of Cr ($Z = 24$, [Ar] $3d^5\,4s^1$) does not violate the Pauli principle.

2 pnts (g) A silicium atom has two p electrons in the outer shell. Which of the possible ${}^{2S+1}L_J$ terms are allowed by the Pauli principle?

1 pnt (h) The neutron decays into a proton, an electron, and an antineutrino: $n \rightarrow p + e^- + \bar{\nu}_e$. Is the antineutrino a boson or a fermion, and why?

Problem 2 (*45 minutes; 15 points in total*)

To both ends of a (massless) rigid rod of length $2a$ a mass M is attached. The midpoint of this rotor is fixed in space. The Hamiltonian is

$$H = \frac{L^2}{2I} ,$$

where $I = 2Ma^2$ is the moment of inertia.

3 pnts (a) What are the eigenstates and corresponding eigenvalues? What is the degeneracy of the energy levels?

4 pnts (b) Assume the rotor is in the state

$$\psi = -i\sqrt{\frac{3}{8\pi}} \sin\theta \sin\phi + \sqrt{\frac{3}{8\pi}} \cos\theta .$$

What are the possible outcomes of a measurement of L^2, and with which probabilities?

2 pnts (c) Answer the same question for a measurement of L_z.

3 pnts (d) Answer the same question for a measurement of L_y. *Hint:* Use the matrix representations given below.

3 pnts (e) The rotor is placed in a weak magnetic field in the z-direction. The Hamiltonian is:

$$H = \frac{L^2}{2I} + \frac{eB}{mc} L_z .$$

Calculate the splitting of the energy levels due to the magnetic field. Sketch the energy spectrum, with and without the magnetic field.

$$Y_0^0 = \sqrt{\frac{1}{4\pi}}, \quad Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos\theta, \quad Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8\pi}} \sin\theta e^{\pm i\phi} .$$

$$L_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad L_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad L_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} .$$

Problem 3 *(35 minutes; 10 points in total)*

An electron, with mass m, is confined in a 3D cubic box with sides of length L, *i.e.* the potential is:

$$\begin{aligned} V(x,y,z) &= 0 \qquad 0 < x,y,z < L\ , \\ &= \infty \qquad x,y,z < 0 \text{ or } x,y,z > L\ . \end{aligned}$$

3 pnts (a) Give the Schrödinger equation. Show that the solution that obeys the proper boundary conditions is

$$\psi(x,y,z) = A \sin(k_x x) \sin(k_y y) \sin(k_z z)\ .$$

What are the conditions on k_x, k_y, and k_z? Give the corresponding energy eigenvalues E. Calculate the normalization constant A (assume that it is real and positive).

2 pnts (b) Discuss the degeneracy of the energy levels.

2 pnts (c) Now put 24 electrons in the box. Assume that they do not interact with each other. What is the lowest possible energy, in units of $\hbar^2\pi^2/(2mL^2)$? What would it be for *spinless* particles with mass m?

3 pnts (d) Consider next 10^{27} electrons (total mass of 1 gram). In order to demonstrate the dramatic effect of the Pauli principle, estimate the mass of this many electrons when they are confined to a cube of side $L = 1$ cm. Assume again that they do not interact with each other. First estimate the energy and then use the formula $M = E/c^2$. What would the answer be for *spinless* particles?

Problem 4 (*35 minutes; 10 points in total*)

A proton is at rest in an eigenstate α_x of S_x, with eigenvalue $+\hbar/2$. At time $t = 0$ it is placed in a magnetic field pointing in the z-direction, $\vec{B} = (0, 0, B)$, in which it is allowed to precess for a time T.

3 pnts (a) Give the relation between the proton spin and its magnetic moment (in terms of the g-factor). Give the time-dependent Schrödinger equation for the spin vector $\xi(t)$. Write the Hamiltonian $H = -\vec{\mu} \cdot \vec{B}$ explicitly as a 2×2 matrix.

4 pnts (b) Solve the equation, taking into account the appropriate boundary condition at $t = 0$. Use the notation of the cyclotron frequency: $\Omega = |e|B/mc$. What is $\xi(T)$?

At time $t = T$ the magnetic field is very rapidly rotated in the y-direction, so that now its components are $(0, B, 0)$. After another time interval T a measurement of S_x is carried out.

3 pnts (c) Give in Dirac notation the probability that the value $+\hbar/2$ will be found, and calculate it.